

---

## Solving a System of Equations

---

### 11-1 Introduction

The previous chapter has shown how to solve an algebraic equation with one variable. However, sometimes there is more than one unknown that must be determined in a group of related equations. This chapter develops methods for finding the values of multiple variables in multiple equations. We begin with a simple example, so you get an idea of where we are headed.

Example - Multiple Unknowns in Multiple Equations

Mary is 5 years older than Joe. Joe's age plus twice Mary's age equals 25.  
How old are Mary and Joe?



Based on the information in the question, we may write the following two equations.

Let Mary's Age =  $M$   
Let Joe's Age =  $J$

$$\begin{aligned}J + 5 &= M \\ J + 2M &= 25\end{aligned}$$

Using one of several different methods developed in this chapter, we find that the only values which make both equations true are  $M = 10$  and  $J = 5$ .

Answer: Mary is 10 years old and Joe is 5

In related equations with multiple variables, it is always assumed that the value of each variable is the same in each equation. For example, if  $x = 3$  and  $y = 2$  for the first equation to be true, then  $x = 3$  and  $y = 2$  in the second equation. Such a group of related equations is called a **system of equations**. Since all equations are true at the same time (the variables have the same values), the equations are also called **simultaneous equations**. A system of equations with at least one solution is called **consistent**. A system of equations with no solutions is called **inconsistent**. (e.g. The system  $x + y = 1$  and  $x + y = 0$  is inconsistent since both equations cannot be true at the same time for any value  $x$  and  $y$ .)



### The System of Equations Golden Rule

*A system of equations may only be solved if there are at least as many independent equations as unknowns.*

#### IMPORTANT RULE!

In other words, one variable may be found with one equation. However, solving for two variables requires at least two independent simultaneous equations. Solving for three variables requires at least three independent simultaneous equations. Before continuing, it is necessary to define an independent equation.

One equation is **independent** from another equation if both cannot take the same form by algebraic manipulation. A system of two equations is given below.

$$2x = y$$

$$x + y = 6$$

It is impossible to algebraically modify one equation to look exactly the same as the other equation. Therefore, the equations are independent. Furthermore, by methods demonstrated in this chapter, we find that the unique solution to the given system of equations is

$$x = 2 \quad \text{and} \quad y = 4$$

Verify that these values are true for both equations in the system.

On the other hand, look at the following system of equations.

$$x + y = 6$$

$$2x + 2y = 12$$

The first equation, when multiplied by 2, is in the exact same form as the second equation. Therefore, these equations are **dependent**. A system of dependent equations has an infinite number of solutions. Observe that the following sets of solutions are all valid for this simple system of equations.

$$x = 0.5 \text{ and } y = 5.5$$

$$x = 1 \text{ and } y = 5$$

$$x = 2 \text{ and } y = 4$$

$$x = 3 \text{ and } y = 3$$

$$x = 4 \text{ and } y = 2$$

... etc

Please verify the following example.

#### Example - Independent or Dependent Equations

Are the following simultaneous equations independent or dependent?

$$3x + 4y + 9 = 8$$

$$3x + 4y = -1$$

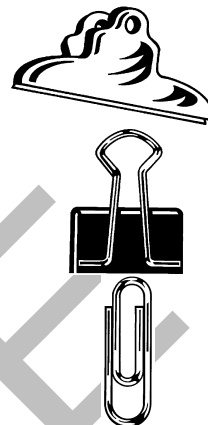
Answer: Dependent

This chapter (in addition to most Algebra texts) will only present techniques and examples for simultaneous linear equations which are consistent. Remember from chapter 10 that quadratics and higher order equations may be factored into linear equations.

The values of variables in a system of independent linear equations may be found by one of the following methods.

1. Substitution of Equations
2. Elimination of Variables
3. Matrix Reduction
4. Cramer's Rule

The first two methods are commonly presented in Algebra I and Algebra II texts. Solving systems using the last two methods is usually first explained in Linear Algebra, but is also presented in this chapter. Every method accomplishes the same purpose -- multiple unknowns are found from a system of simultaneous equations. Different methods are easier to use depending on the problem. That's why so many different methods are shown.



THEY DO THE  
SAME THING, BUT  
IN DIFFERENT  
WAYS

## 11-2 Substitution of Equations

From topics learned in chapter 10, we can solve for one unknown in any one algebraic equation. Based on this information, if we can reduce a system of equations with multiple unknowns into one equation with one unknown, the value of a single variable may be found. Once the value of a single variable is found, the values of other variables may be found. This is the goal of the substitution of equations method.

Steps for using this method on a system of two equations are given below. Steps may be slightly modified for a system of more than two equations. (Just like steps for adding two numbers may be slightly modified to add three numbers.)

### STEP - BY - STEP

#### Solving a System of Two Equations Using Substitution

1. Isolate for either variable in one equation.
2. Replace the isolated variable (from step #1) in the other equation.
3. Solve for the only variable in the equation from step #2.
4. Solve for the second variable using either equation and the result from step #3.

Let's clarify the steps with an example.

Example - Solving Simultaneous Equations with Substitution

Solve the following system of two equations.

$$3x + 4y + 9 = 8$$

$$x + y = 0$$

Step 1: Isolate for either variable in one equation. It is simple to isolate for either  $x$  or  $y$  in the second equation. Let's choose  $x$  just for fun.

$$x = -y$$

Step 2:  $x$  was isolated in step #1, and found to equal  $-y$ . Therefore, we can now replace  $x$  with  $-y$  in the first equation.

$$3(-y) + 4y + 9 = 8$$

Step 3:  $y$  is the only variable in the equation, and therefore the equation may be solved for  $y$ .

$$3(-y) + 4y + 9 = 8$$

$$y + 9 = 8$$

$$y = -1$$

Step 4: Solve for  $x$  by substituting  $y = -1$  in either original equation since both equations are true. The second equation is less complicated, so we'll use the second equation.

$$x + (-1) = 0$$

$$x = 1$$

Therefore, the solutions are  $y = -1$  and  $x = 1$ . Check the answers by inserting the values into each equation, and verifying that both equations are true or 'satisfied'.

$$3(1) + 4(-1) + 9 = 8$$

$$8 = 8 \quad \checkmark \text{ yes}$$

$$1 + (-1) = 0$$

$$0 = 0 \quad \checkmark \text{ yes}$$

Answers:  $x = 1$  and  $y = -1$

Let's solve the same system again, except isolate for  $x$  in the first equation rather than the second. We had better get the same answer!

Step 1: Isolating for  $x$  yields,

$$3x + 4y + 9 = 8$$

$$3x + 4y = -1$$

$$3x = -1 - 4y$$

$$x = \frac{-1 - 4y}{3}$$

Step 2: Replace  $x$  in the second equation to get one equation with one unknown.

$$\frac{-1 - 4y}{3} + y = 0$$

Step 3: Solve for  $y$ .

$$-1 - 4y + 3y = 0$$

$$-1 = y$$

Step 4: Solve for  $x$  by substituting  $y = -1$  in either equation.

$$x + (-1) = 0$$

$$x = 1$$

This is the same result obtained earlier. It is helpful to isolate and substitute in the equations which make the problem easier to solve.

Let's try substitution with two dependent equations and see what happens. Look at the following system of dependent equations.

$$\begin{aligned}x + y &= 3 \\ 3x + 3y &= 9\end{aligned}$$

Step 1: Using the first equation,

$$y = 3 - x$$

Step 2: Substituting for  $y$  in the second equation yields

$$3x + 3(3 - x) = 9$$

$$3x + 9 - 3x = 9$$

$$9 = 9$$

$9 = 9$  is obvious, but nothing has been solved.  $x$  was canceled and no variables remain. The problem is that the equations are dependent, so an infinite number of  $x$  and  $y$  combinations are valid. A system of dependent equations will always cancel the variable and produce a result as shown in step #2 (i.e.  $a = a$ ).

Verify the following examples using the substitution of equations method. You should be able to solve a system of three (or more) equations using this method. Substitute variables to obtain one equation and one unknown. Then solve, and use the result to find the value of other variables.

Examples - Solving Systems of Equations with Substitution

Solve the following system.  
(Hint: Substitute for 2 variables in the final equation)

$$\begin{aligned}x + y &= 5 \\z &= 3 - x \\x + y + z &= 9\end{aligned}$$

Answer:  $x = -1, y = 6, z = 4$

### 11-3 Elimination of Variables

The goal of this method is to generate one new equation with one unknown from the system of equations. The new equation is created by subtracting (or adding) the existing equations in the system. The equations must be carefully subtracted (or added) such that the resulting new equation only contains one variable.

Steps for solution are given below for a system of two equations. Again, steps may be slightly modified for a system with more than two equations.

**STEP - BY - STEP**

**Solving a System of Two Equations Using Elimination**

1. Algebraically modify one equation (or both equations), such that the magnitude of a variable's coefficient is the same in both equations.
2. Subtract (or add) the two equations to create a new equation with only one variable.
3. Solve for the variable in the newly generated equation.
4. Solve for the second variable using an original equation and the result from step #3.

Let's clarify the steps with an example.

Example - Solving Simultaneous Equations with Elimination

Solve the following system of two equations.

$$\begin{aligned}3x + 4y + 9 &= 8 \\x + y &= 0\end{aligned}$$

Step 1: Let's modify the second equation by multiplying it by 3. Then the coefficient of  $x$  will be the same in both equations.

Remember from chapter 10 that if the left side of the equation is multiplied by three, the right side of the equation must also be multiplied by three.

$$3(x + y) = 3(0)$$

$$3x + 3y = 0$$

Step 2: Subtract equations so that the variable  $x$  cancels in the result. (Adding the equations would not produce a new equation with only one variable.)

$$\begin{array}{r} 3x + 4y + 9 = 8 \\ - (3x + 3y = 0) \\ \hline \end{array}$$

The subtraction is more clear if all coefficients are aligned. Therefore, move 9 to the right side of the first equation to align the constants. Remember that the entire equation must be subtracted from the other, not just the term  $3x$ .

$$\begin{array}{r} 3x + 4y = -1 \\ - (3x + 3y = 0) \\ \hline 0x + y = -1 \end{array}$$

Step 3: The subtraction produced one equation with one unknown. In this example, the variable has already been isolated.

$$y = -1$$

Step 4: Solve for  $x$  by substituting  $y = -1$  in one of the original equations. We'll choose the second.

$$x + (-1) = 0$$

$$x = 1$$

$$\text{Answers: } y = -1, x = 1$$

This is the same example and same solution as that obtained in the previous section.

*Why does this method work?*

This approach applies rules demonstrated in section **10-4**. As you know from that section, if one value is added to one side of the equation, the same value must be added to the other side of the equation to keep the equation true.

In the elimination of variables method the exact same process is repeated. The only twist is that different forms of a particular value are subtracted (or added) from each side of the equation. In the above example, we were given

$$x + y = 0$$

Therefore, subtracting  $(x + y)$  is the same as subtracting 0. Elimination uses this technique to eliminate variables. The newly generated equation is simply a modification of an original equation.

We have stated that equations may be subtracted or added to eliminate a variable. Look at the following system of equations. Note that the magnitude of the coefficient of  $x$  is the same in both equations (i.e. 3).

$$3x + 4y + 9 = 8$$

$$-3x - 3y = 0$$

In this example, the equations may be added rather than subtracted to eliminate the variable  $x$ . This example is actually the same system as that presented earlier, and results in  $x = 1$ ,  $y = -1$ .

Verify the following example using elimination of variables.

#### Example - Solving Simultaneous Equations with Elimination

Solve the following system of two equations using elimination.

$$\begin{aligned} 2x - 3y &= -7 \\ x + 6y &= 34 \end{aligned}$$

$$\text{Answer: } y = 5, x = 4$$

If subtracting (or adding) equations results in a newly developed equation such as  $0 = 0$ , no variables may be solved. The equations in such a system are dependent, as shown in the substitution method. If subtracting (or adding) equations results in a newly developed equation such as  $0 = 1$ , or another false mathematical statement, the system is inconsistent, and no solution exists.



**HELPFUL TIP**



### **WAIT !!**

The remaining sections of this chapter are advanced and show two optional methods of solving simultaneous equations. You may skip the remainder of this chapter if you do not need (or want) to know how to use matrices. Matrices are not used elsewhere in this text.

## **11-4 Using Matrices**

It becomes difficult to use either the substitution or the elimination method if the system contains more than two equations. One problem is simply writing all of the equations with the proper signs and variables. Mistakes can easily be made in merely copying equations.

Upon reviewing the elimination method, we realize that the values of the coefficients are the important aspects since each equation in the system uses the same variables. (Remember, the coefficient of a variable not listed is simply zero.) Therefore, a shortcut notation for writing the system of equations has been developed, a matrix.



We begin with some important definitions. Examples of each will be given shortly.

A **matrix** is a rectangular array of numbers. In general, the **size** of a matrix is referred to as “# rows by # columns.” An **augmented matrix** is a matrix which contains all of the coefficients in a set of linear simultaneous equations, and the constants equal to the linear equations. A matrix containing only variable coefficients (no constants) is a **coefficient matrix**.

A matrix with the same number of columns as rows is called a **square** matrix. In all of our examples, we will assume that every system of equations has the same number of variables as equations (which is a necessary condition to solve a system of equations). Therefore, every coefficient matrix developed in this chapter will be square.

Each number in a matrix is called an **element**. In general, we may refer to each element as

$$a_{r,c}$$

where  $r$  represents the row in the matrix, and  $c$  represents the column in the matrix. You may also remember the element as  $a_{\text{row,column}}$  or  $a_{\text{down,across}}$ .

Let's clarify some of the definitions with an example.

#### Example - Writing Matrices

Write an augmented matrix for the following system of equations.

What is element  $a_{3,4}$  in the augmented matrix?

Write a coefficient matrix for the system of equations.

$$\begin{aligned}x + y + 2z &= 9 \\2x + 4y - 3z &= 1 \\3x + 6y - 5z &= 0\end{aligned}$$

An augmented matrix is written using the coefficients of each variable and constants right of the equals sign. The size of the augmented matrix given below is 3 by 4 or equivalently  $3 \times 4$ .

Answer: The augmented matrix is:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

The element in row 3, column 4 is 0.

Answer:  $a_{3,4} = 0$

The coefficient matrix is the exact same as the augmented matrix except with the column containing constants removed.

Answer: The coefficient matrix is:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

All coefficients of the same variable must be aligned in a single column. The same is true for constants which must be isolated on the opposite side of the equals sign. Therefore, the first step in writing a matrix is to rewrite the equations with the same variables aligned in each equation. Constants should all be placed on the right side of each equation.

You should understand that the order of the rows is not important. Usually, the rows are simply listed in the same order as the equations are given in the original problem.

Therefore, the system of equations from the previous example may equivalently be rewritten as shown.

$$\begin{aligned}4y + 2x - 3z &= 1 \\y + x + 2z &= 9 \\6y + 3x - 5z &= 0\end{aligned}$$

and accurately described with the following augmented matrix,

$$\left[ \begin{array}{cccc} 4 & 2 & -3 & 1 \\ 1 & 1 & 2 & 9 \\ 6 & 3 & -5 & 0 \end{array} \right]$$

We can denote some of the elements in the above matrix as follows:

$$a_{1,1} = 4 \quad a_{1,2} = 2 \quad a_{2,1} = 1 \quad a_{2,4} = 9 \quad a_{3,4} = 0$$

Each row in the matrix is simply the constant coefficients of one equation in the system of equations. Therefore, any operation allowed on an equation is also allowed on a matrix row.

#### Allowed Augmented Matrix Manipulations

- Any row may be multiplied by a non-zero constant.
- Any two rows may be interchanged.
- A multiple of one row may be added to another row.

Note that columns in a matrix cannot be interchanged. (The matrix will eventually provide the solutions for the system, so interchanging columns results in swapped answers for variables.)

In our discussion of using matrices to solve simultaneous equations, there is no need to know how to add, subtract, or multiply matrices. However, knowing how to perform these operations may be helpful in some other area of your education. Therefore, a brief explanation of each is given prior to describing how to solve systems of equations using matrices. The remainder of this section may be skipped if you are only trying to learn how to solve simultaneous equations using matrices.

To add two matrices, each matrix must be the same size. Add elements corresponding to the same row and column of each matrix.

Example - Adding Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 5 \\ 7 & 9 & 11 \\ 3 & 11 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 8 \\ 11 & 14 & 17 \\ 10 & 19 & 13 \end{bmatrix}$$

Subtracting matrices also consists of subtracting corresponding elements.

A constant number is multiplied by a matrix by multiplying each element in the matrix by the constant. This should be clear, since it is the same as multiplying each equation in the system by the same constant.

Multiplying matrices is much more complicated.

**STEP - BY - STEP**

**Multiplying matrices**

Assume the problem consists of two matrices **A** (size  $r_1 \times c_1$ ), and **B** (size  $r_2 \times c_2$ ). The product **AB** may only be obtained if the number of columns in matrix **A** is equal to the number of rows in matrix **B** (i.e.  $c_1 = r_2$ ). The resulting matrix will be size  $r_1 \times c_2$ .

When multiplying matrices, **AB**  $\neq$  **BA**, so the *Commutative Law of Multiplication does not apply to matrices*.

1. To obtain element  $a_{r,c}$  in **AB**, multiply each element in row  $r$  of **A** by each corresponding element in column  $c$  of **B**, and add the resulting products.
2. Repeat step #1 for all elements.

At first, multiplying matrices is quite confusing. Row 2 of the first matrix and column 3 of the second matrix are combined as shown to obtain the element in row 2, column 3 of the product.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 26 & \square \end{bmatrix}$$

$$2(4) + 6(3) + 0(5) = 26$$

Multiplying each row of the first matrix with every column of the second matrix produces the completed matrix as given below.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

It is not possible to divide matrices.

### 11-5 Solving Systems with Matrix Reduction

Solving a system of equations with matrix reduction is basically the same as the elimination of variables. The only difference is that equations are written in matrix form, using only coefficients.

Assume that a system of equations with variables  $x$ ,  $y$ , and  $z$  is described by the following augmented matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 12 \end{bmatrix}$$

This matrix immediately provides answers to the values of each variable, as shown by writing the three equations represented by the matrix.

$$\begin{aligned} x + 0y + 0z &= 8 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 12 \end{aligned}$$

or

$$\begin{aligned} x &= 8 \\ y &= 4 \\ z &= 12 \end{aligned}$$

The above matrix form is called **reduced row-echelon** form. All variable values can be determined by looking at the augmented matrix since each row contains only a single coefficient of 1, in addition to the answer in the rightmost column.

Alternately, the following matrix also provides information from which variable values may be determined.

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{12} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Using the last row, we can immediately write:

$$z = 3$$

The equation from the middle row may be rewritten

$$y - \frac{7}{2}z = -\frac{17}{12}$$

Substituting  $z = 3$  in the above equation yields  $y = 2$ .

Verify that  $x = 1$  by substituting values for  $y$  and  $z$  into the equation determined from the top row.

The above matrix is in **row-echelon** form, which means that coefficients of 1 are on a diagonal in the matrix. The ones serve as a boundary to a group of all zeros in the matrix.

**STEP - BY - STEP****Solving a System of Equations with Matrix Reduction**

1. Write the appropriate augmented matrix.
2. Apply any of the allowable augmented matrix manipulations to get the matrix in either reduced row-echelon form, or row-echelon form.
3. Solve the variables using the newly developed matrix coefficients.

Example - Using Row Reduction to Solve a System of Equations

Solve for variables  $x$ ,  $y$  and  $z$  in the following augmented matrix using row reduction.

$$\left[ \begin{array}{cccc} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array} \right]$$

Step 1: Add  $(-2)$  times the first row to the second row to make the first coefficient in row two 0.

$$\left[ \begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array} \right]$$

Step 2: Add  $(-3)$  times the first row to the third row to make the first coefficient in row three 0.

$$\left[ \begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right]$$

Step 3: Multiply the second row by  $1/2$  to make the second coefficient 1.

$$\left[ \begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array} \right]$$

Step 4: Add  $(-3)$  times the second row to the third row to make the second coefficient in row three equal 0.

$$\left[ \begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array} \right]$$

Step 5: Multiply the third row by -2 to make the third coefficient in row three 1.

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The matrix is now in row-echelon form, and variables may be solved. However, let's continue reduction to achieve reduced row-echelon form.

Step 6: Add (-1) times the second row to the first row to make the second coefficient in row one equal to 0.

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 7: Add  $-11/2$  times the third row to the first row, and  $7/2$  times the third row to the second row to obtain reduced row echelon form. (Verify that you can obtain the matrix below.)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Therefore,

Answer:  $x = 1 \quad y = 2 \quad z = 3$

By studying the example above, you will see patterns to reducing a matrix to the desired form. No one said that solving systems of equations was simple. However, with practice you will find that solving systems of equations using matrices is not difficult. Rather, it is often just a long process. You should understand *why* solving equations with matrices works. It is merely the repeated application of the elimination of variables, using only coefficients.

Verify the following example.

Example - Solving Systems of Equations with Row Reduction

Solve the following system of equations using matrices and row reduction.

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 5y + 3z &= 3 \\ 1x \quad + 8z &= 17 \end{aligned}$$

Answer:  $x = 1, y = -1, z = 2$

### 11-6 Solving Systems with Cramer's Rule

Prior to demonstrating how to use Cramer's rule, more definitions are required. Assume a square matrix  $\mathbf{A}$  exists. The **determinant** of  $\mathbf{A}$  is the sum of the products of the top-left to bottom-right diagonal elements of  $\mathbf{A}$  minus the sum of the products of the top-right to bottom-left diagonal elements of  $\mathbf{A}$ . It may be referred to as  $\det(\mathbf{A})$ . Bars around the matrix represent the matrix's determinant (similar to absolute value symbols). The determinant of a matrix is a number, not another matrix.

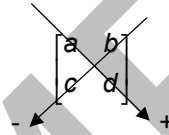


UNDERSTANDING THIS SECTION WILL TAKE A LOT OF TIME.

Example - Determinant of a  $2 \times 2$  Matrix

State the determinant of  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Arrows will assist in clarification of the definition.

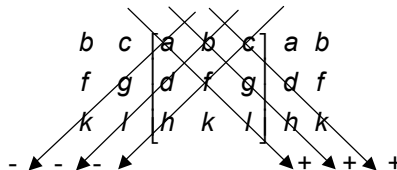


Answer:  $ad - bc$

Example - Determinant of a  $3 \times 3$  Matrix

Find:  $\begin{vmatrix} a & b & c \\ d & f & g \\ h & k & l \end{vmatrix}$

Arrows and repeating columns where the arrows 'wrap around the matrix' to make the necessary completed diagonals help clarify. Note that a  $2 \times 2$  did not require 'wrapping columns' to complete diagonals. Can you see the completed diagonals without repeating the columns outside the matrix?



Answer:  $afh + bgk + cdh - (cfh + bdl + agk)$

The determinants of  $4 \times 4$ ,  $5 \times 5$  and larger matrices all require 'wrapping around columns' to complete the diagonals and follow the example shown for a  $3 \times 3$  matrix.

Finding the determinant using this method can get confusing. An alternative method to obtaining the determinant is given below. (Although initially you may find it equally or even more confusing!) Sorry, but more definitions are required.

The **minor of element**  $a_{r,c}$  is the determinant of the matrix that remains after the  $r^{\text{th}}$  row and  $c^{\text{th}}$  column are removed. Let's represent the minor with  $M_{r,c}$ . Let  $C_{r,c} = (-1)^{r+c} M_{r,c}$  which is the **cofactor of**  $a_{r,c}$ . Again, examples will help clarify all of these definitions.

Example - Finding Minors of Matrices

Find the minor and cofactor of  $a_{3,2}$  of the following matrix:

$$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

First, simply remove row three and column two from the square matrix.

$$\begin{bmatrix} 3 & & -4 \\ 2 & & 6 \\ & & \end{bmatrix}$$

The resulting matrix follows:

$$\begin{bmatrix} 3 & -4 \\ 2 & 6 \end{bmatrix}$$

$M_{3,2}$  is the determinant of the above matrix.

$$M_{3,2} = 3(6) - 2(-4) = 18 + 8 = 26$$

The cofactor is  $C_{3,2} = (-1)^{3+2} M_{3,2}$

$$C_{3,2} = (-1)^5(26) = -26$$

Answer:  $M_{3,2} = 26$       $C_{3,2} = -26$

From the definitions, note that the magnitude of the cofactor and minor are always equal. If the row and column are both odd or both even, the exponent of  $(-1)$  is even, and the cofactor equals the minor. However,  $C_{r,c} = (-1) \cdot M_{r,c}$  if either the row or column number is odd.

With these new definitions, we can establish a new method for finding the determinant of a matrix. Remember, the goal is to solve simultaneous equations. We'll get to that shortly. Using determinants is merely one step in using Cramer's Rule.



**STEP - BY - STEP****Finding the Matrix's Determinant Using Cofactor Expansion**

For any square matrix **A**,

1. Choose any row or column in matrix **A**.
2. Multiply each element in the chosen row or column by the cofactor of the element.
3. Add all of the intermediate results in step #2 to produce the determinant.

Let's clarify with an example.

Example - Finding the Determinant

Find the determinant of the following matrix. First, use only multiplication of diagonal elements. Second, use cofactor expansion to verify the answer.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Multiplying appropriate diagonals provides the following answer.

$$\begin{aligned} \det(\mathbf{A}) &= 3(-4)(-2) + (1)(3)(5) + 0(-2)(4) - 0(-4)(5) - 1(-2)(-2) - (3)(3)(4) \\ &= 24 + 15 - 4 - 36 \\ &= -1 \end{aligned}$$

Using cofactor expansion along the top row yields:

$$\det(\mathbf{A}) = 3(-1)^2 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0(-1)^{1+3} \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$\det(\mathbf{A}) = 3(1)(8 - 12) + 1(-1)(4 - 15) + 0(1)(-8 + 20)$$

$$\det(\mathbf{A}) = 3(-4) - 1(-11) + 0$$

$$\det(\mathbf{A}) = -1$$

$$\text{Answer: } \det(\mathbf{A}) = -1$$

Cofactor expansion along any row or column yields the same result. Verify the above example by performing cofactor expansion along a different row or column.

Let's pause and summarize what we already know about matrices. We've learned how to:

1. Setup a system of equations as an augmented or coefficient matrix.
2. Find the determinant of a square matrix by multiplying diagonals.
3. Find the determinant of a square matrix by cofactor expansion.

We are finally able to solve a system of equations using Cramer's Rule. A formal definition and proof of Cramer's Rule is not included due to mathematical complexity. However, the following steps are required to implement Cramer's Rule.

### STEP - BY - STEP

#### Using Cramer's Rule to Solve a System of Equations

Assume the system of equations consists of variables  $x_1, x_2, \dots, x_n$  where  $n \geq 2$

1. Setup coefficient matrix **A** and a 1-column matrix called **B**. **B** consists of the column of numbers that would be included in the augmented matrix, but not in the coefficient matrix.
2. Form matrix **A**<sub>1</sub> by replacing column 1 in **A** with **B**, form matrix **A**<sub>2</sub> by replacing column 2 in **A** with **B**, repeat for every matrix **A**<sub>n</sub>.
3. Find determinants of all **A** matrices.
4. Solve for the desired variable  $x_n$  where  $x_n = \det(\mathbf{A}_n) / \det(\mathbf{A})$

Again, the process is long but not mathematically difficult. Determinants may be found by using cofactor expansions, or by summing appropriate diagonals. Use whichever method you prefer. However, as matrices become larger, using cofactor expansions is much less work than multiplying and summing diagonals. An example will clarify the steps of solving equations with Cramer's Rule.

#### Example - Using Cramer's Rule to Solve a System of Equations

Solve the following system of equations using Cramer's Rule

$$\begin{aligned} x + 2z &= 6 \\ -3x + 4y + 6z &= 30 \\ -x - 2y + 3z &= 8 \end{aligned}$$

Steps 1 and 2: We need to solve for  $x, y,$  and  $z$ . Therefore, we'll first setup all of the required matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$$

Step 3: Find the determinants of all matrices. (The reader should verify that these are correct.)

$$\det(\mathbf{A}) = 44 \quad \det(\mathbf{A}_1) = -40 \quad \det(\mathbf{A}_2) = 72 \quad \det(\mathbf{A}_3) = 152$$

Step 4: Solve for all variables.

$$x = \det(\mathbf{A}_1) / \det(\mathbf{A}) = -40/44 = -10/11$$

$$y = \det(\mathbf{A}_2) / \det(\mathbf{A}) = 72/44 = 18/11$$

$$z = \det(\mathbf{A}_3) / \det(\mathbf{A}) = 152/44 = 38/11$$

$$\text{Answers: } x = -\frac{10}{11} \quad y = \frac{18}{11} \quad z = \frac{38}{11}$$

Verify the following example.

Example - Using Cramer's Rule to Solve a System of Equations

Solve the following system of equations using Cramer's Rule

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 5y + 3z &= 3 \\ 1x + \quad + 8z &= 17 \end{aligned}$$

$$\text{Answer: } x = 1, y = -1, z = 2$$

## 11-7 Summary

You have learned how to solve systems of equations using the following methods:

1. Substitution of Equations
2. Elimination of Variables
3. Matrix Reduction
4. Cramer's Rule

You should understand *why* the first three methods presented work. Due to mathematical complexity, discussion of Cramer's Rule did not show why this method works. However, Cramer's rule is very useful and is easily programmed on computers.

Substitution and elimination are generally used if there are only two or three equations in a system. Matrices are generally used if there are three or more matrices in the system. Only an introduction of matrices was presented in this chapter. The course Linear Algebra (usually first taught in college) expands on the introductory ideas presented in this chapter to solve a variety of problems containing simultaneous equations.

Solving systems of equations is important. Further understanding will be obtained by reading subsequent chapters.

**Important Definitions**

Simultaneous Equations

Consistent System of Equations

Inconsistent System of Equations

Independent Equations

Dependent Equations

Matrix

Matrix Size

Augmented Matrix

Coefficient Matrix

Reduced Row-Echelon Form

Row-Echelon Form

Determinant

Minor

Cofactor

**You should be able to ...**

1. Determine if equations are independent or dependent
2. Use Substitution of Equations to solve a system of linear equations
3. Use Elimination of Variables to solve a system of linear equations
4. Setup an augmented or coefficient matrix if given a system of equations
5. Add, subtract, or multiply two matrices
6. Solve a system of linear equations with Matrix Reduction to either row-echelon form or reduced row-echelon form
7. Find the determinant of a matrix by multiplying diagonals, or using Cofactor Expansion
8. Solve a system of equations using Cramer's Rule